XVI. *The Binomial Theorem demonstrated by the Principles of Multiplication.* *By* Abram Robertson, *A. M. of* Christ Church, Oxford; *F. R. S.* *In a Letter to the Rev. Dr.* Maskelyne, *F. R. S. and Astronomer Royal.*

Read May 21, 1795.

REV. SIR, Christ Church, Oxford, Oct. 27th, 1794.

A CONSIDERATION of the very high importance and extensive utility of the binomial theorem, having induced me to enter upon an examination of the methods in which, at different times, it has been demonstrated; and having frequently reviewed them, and deliberated with myself upon the subject, I was convinced that a demonstration begun and conducted upon the obvious principles of multiplication was still wanted, much to be desired, and also attainable. For to these principles involution must be ultimately referred, in whatever form it may be presented; and it therefore appeared, that an investigation of the theorem effected by them only, was likely to be as simple and perspicuous as the subject will permit.

I think it needless to enter into a minute account of the demonstrations heretofore published, or to enumerate the objections which have been or may be made to them. It is well known to mathematicians that they are effected either by induction, by the summation of figurate numbers, by the doctrine of combinations, by assumed series, or by fluxions: but that multiplication is a more direct way to the establishment of the theorem than any of these, cannot, I suppose, be doubted. Proceeding by it, we have always an evident first principle in view, to which, without the aid of any doctrine foreign to the subject, we can appeal for the truth of our assertions, and the certainty and extent of our conclusions.

The following demonstration, which owes its origin to the abovementioned train of thinking, might be divided into two parts; but I thought it more advisable to divide it into articles, and number them for the sake of references. That which might be called the first part, extends from the first to the end of the twelfth article, and contains the investigation of the theorem, as far as it relates to the raising of integral powers. The remaining articles constitute the second part, which contains the demonstration of the theorem as applicable to the extraction of roots, or the raising of powers, when the exponents are vulgar fractions. If the assumption of the series, in which the theorem is usually expressed, be allowed, the first part might be inferred as a corollary from the demonstration of the second. For having proved that $\overline{x+z}^{\frac{n}{r}} = x^{\frac{n}{r}} + \frac{n}{r} z x^{\frac{n}{r}-1} + \frac{n}{r} \times \frac{\frac{n}{r}-1}{2} z^2 x^{\frac{n}{r}-2} +$, &c. it follows, that when r is equal to 1, then $\overline{x+z}^n = x^n + n z x^{n-1} + n \times \frac{n-1}{2} \times z^2 x^{n-2} +$, &c. I could not, however, think of suppressing the first part, as the binomial series is so easily investigated in it from first principles.

Upon examining the Philosophical Transactions, I found a demonstration of this important theorem by CASTILLIONEUS, in the XLIId Volume. In effecting it he had recourse to the doctrine of combinations of quantities, in that part of his investigation which relates to the raising of integral powers; and by extending this to the involution of a multinomial, and employing an assumed series, he made out the most general case, or that in which the exponent is a fraction. In neither of the cases, however, in my opinion, is the law of continuation proved with sufficient perspicuity. In the XLVIIth Vol. of the Transactions there is a paper, not expressly on the binomial theorem, by the celebrated Mr. THOMAS SIMPSON, in which the case for raising integral powers is demonstrated by fluxions.

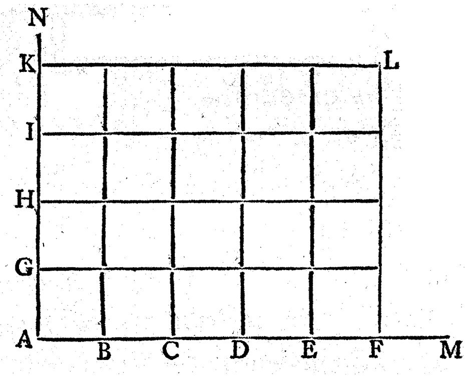
With respect to the following demonstration, I submit it to your inspection, with the most perfect confidence in your judgment and candour; and if it appears to you not unworthy of the attention of the Royal Society, by presenting it to that learned body you will add to the favours which you have already conferred upon me.

I am, &c.

A. ROBERTSON.

1. The product arising from the multiplication of any number of quantities * into one another, continues the same in value, in every variation which may be made in the arrangement of the quantities which compose it. Thus $p \times q \times r \times s = pqrs = spqr = psqr = pqsr =$ any other arrangement of the same quantities.

* When I speak of the multiplication of quantities into one another, I mean the multiplication of the numbers into one another which measure those quantities.



For let AM, AN be two indefinite straight lines at right angles to one another, and in AM set off AB, BC, CD, DE, EF, &c. equal to one another, and in number equal to the number of units in the quantity p*; and in AN set off AG, GH, HI, IK, &c. each equal to AB, and let the number of these parts be equal to the number of units in the quantity q. Complete the rectangle KF, and draw straight lines parallel to AK, through the points B, C, D, E, and let them meet the opposite side KL of the parallelogram. Through the points G, H, I, draw straight lines parallel to AF, and let them meet FL, the opposite side of the parallelogram. Then will the whole rectangle KF be divided into squares, each equal to GB. Now when p is multiplied into q, the number of units in the product is equal to the number of units in p repeated as often as there are units in q. But the number of squares in the rectangle KF is equal to the number of parts in AF repeated as often as there are parts in AK; and therefore, by the above construction, the number of squares in the rectangle KF is equal to the number of units in p repeated as often as there are units in q. Hence the number of squares in the rectangle KF is equal to the number of units in $p \times q$. In the same manner it may be proved that the number

* When I speak of the number of units in a quantity, I mean the number of units in the number measuring that quantity.

of squares in the rectangle K F is equal to the number of units $q \times p$; and consequently $pq = qp$.

Hence it follows that $pqrs = spqr$; for by the above, $pqr \times s = s \times pqr$. Also $spqr$ is equal to $psqr$; for $spqr = sp \times q \times r =$ (by the above) $ps \times q \times r$. Again, $psqr = pqsr$; for $psqr = p \times sqr = p \times qsr = pqsr$, by the above. And if $x + a = p$, $x + b = q$, $x + c = r$, $x + d = s$, $x + e = t$, &c. then $\overline{x+a} \times \overline{x+b} \times \overline{x+c} \times \overline{x+d} \times \overline{x+e} = pqrst = \overline{x+a} \times \overline{x+b} \times \overline{x+c} \times \overline{x+e} \times \overline{x+d} = pqrts =$ any other arrangement which can take place in the quantities.

2. It is evident that each of the quantities a, b, c, &c. will be found the same number of times in the compound product arising from $\overline{x+a} \times \overline{x+b} \times \overline{x+c} \times \overline{x+d} \times \overline{x+e}$, &c. For this product is equal to $pqrst = pqrs \times \overline{x+e} = pqrt \times \overline{x+d} = pqst \times \overline{x+c} = prst \times \overline{x+b} = qrst \times \overline{x+a}$, by substituting for the compound quantities, $x + a$, $x + b$, &c. their equals p, q, &c. Wherefore, in the compound product, each of the quantities a, b, c, &c. will be found multiplied into the products of all the others.

3. These things being premised, we may proceed to the multiplication of the compound quantities $x + a$, $x + b$, $x + c$, &c. into one another; and in order to be as clear as possible in what follows, let us consider the sum of the quantities, a, b, c, &c. or the sum of any number of them multiplied into one another, as coefficients to the several powers of x, which arise in the multiplication. By considering products which contain the same number of the quantities a, b, c, &c. as homologous, the

multiplication will appear as follows, and equations of various dimensions will arise, according to the powers of x.

$$\begin{array}{l} x + a = p \\ x + b = q \\ \hline \end{array}$$

$$x^2 \left.\begin{matrix} + a \\ + b \end{matrix}\right\} x + ab = pq;$$ a quadratic equation.

$$\begin{array}{l} x + c = r \\ \hline \end{array}$$

$$x^3 \left.\begin{matrix} + a \\ + b \\ + c \end{matrix}\right\} x^2 \left.\begin{matrix} + ab \\ + ac \\ + bc \end{matrix}\right\} x + abc = pqr;$$ a cubic.

$$\begin{array}{l} x + d = s \\ \hline \end{array}$$

$$x^4 \left.\begin{matrix} + a \\ + b \\ + c \\ + d \end{matrix}\right\} x^3 \left.\begin{matrix} + ab \\ + ac \\ + bc \\ + ad \\ + bd \\ + cd \end{matrix}\right\} x^2 \left.\begin{matrix} + abc \\ + abd \\ + acd \\ + bcd \end{matrix}\right\} x + abcd = pqrs;$$ a biquadratic.

$$\begin{array}{l} x + e = t \\ \hline \end{array}$$

$$x^5 \left.\begin{matrix} + a \\ + b \\ + c \\ + d \\ + e \end{matrix}\right\} x^4 \left.\begin{matrix} + ab \\ + ac \\ + bc \\ + ad \\ + bd \\ + cd \\ + ae \\ + be \\ + ce \\ + de \end{matrix}\right\} x^3 \left.\begin{matrix} + abc \\ + abd \\ + acd \\ + bcd \\ + abe \\ + ace \\ + bce \\ + ade \\ + bde \\ + cde \end{matrix}\right\} x^2 \left.\begin{matrix} + abcd \\ + abce \\ + abde \\ + acde \\ + bcde \end{matrix}\right\} x + abcde = pqrst;$$ a sursolid.

&c.

4. From the above it appears, that the coefficient of the highest power of x in any equation is 1; but the coefficient of any other power of x in the same equation consists of a certain number of members, each of which contains one, two, three, &c.

of the quantities of a, b, c, &c. Thus the coefficient of the third term of any equation, is made up of members, each of which contains two of the quantities only, as, $ab + ac + bc$, the coefficient of the third term in the cubic equation. And indeed, not only from inspection, but also from considering the manner in which the equations are generated, it is evident, that each member of any coefficient has as many of the quantities in it, as there are terms in the equation preceding the term to which it belongs. Thus, $abc + abd + acd + bcd$ is the coefficient of the fourth term in the biquadratic, each of the members has three quantities in it, and three terms precede that to which they belong.

5. When any equation is multiplied in order to produce the equation next above it, it is evident that the multiplication by x produces a part in the equation to be obtained, which has the same coefficients as the equation multiplied. Thus, multiplying the cubic equation by x we obtain that part of the biquadratic which has the same coefficients as the cubic: the only effect of this multiplication being the increase of the exponents of x by 1.

6. But when the same equation is multiplied by the quantity adjoined to x by the sign $+$, each term of the product, in order to rank under the same power of x, must be drawn one term back. Thus when the first term of the cubic is multiplied by d, the product must be placed in the second term of the biquadratic. When the second term of the cubic is multiplied by d, the product must be placed in the third term of the biquadratic: and so of others.

7. As the equation last produced is the product of all the compound quantities $x + a$, $x + b$, $x + c$, &c. into one ano-

ther, and as it was proved in the second article that each of the quantities a, b, c, &c. must be found the same number of times in this product, if we can compute the number of times any one of those quantities enters into the coefficient of any term of the last equation, we shall then know how often each of the other enters into the same coefficient: and this may be done with ease, if of the quantities a, b, c, &c. we fix upon that used in the last multiplication. For the last equation, and indeed any other, may be considered as made up of two parts; the first part being the equation immediately before the last multiplied by x, according to the 5th article, and the other being the same equation multiplied by the quantity adjoined to x by the sign $+$, last used in the multiplication, according to the 6th article. This last used quantity, therefore, never enters into the members of the coefficient of the first of these two parts, but it enters into all the members of the coefficients of the last of them. But that part into which it does not enter has the same members as the coefficients of the equation immediately before the last, by the 5th article; and when the members of the first part are multiplied by the last used quantity, the product becomes the second part of the whole coefficient above mentioned.

Thus the first part of the cubic equation, by the 5th article is, $x^3 \left.\begin{matrix} + a \\ + b \end{matrix}\right\} x^2 + a\,b\,x$, and as these coefficients are the same as the coefficients in the quadratic equation, being multiplied by c, and arranged according to the 6th article, we have the coefficients of the second part of the cubic, viz. $c \begin{matrix} + a\,c \\ + b\,c \end{matrix} + a\,b\,c$.

Hence it is evident, that there are as many members in any coefficient, which have the last used quantity in them, as there are members in the coefficient preceding, which have not the same quantity; and as it has been proved that each of the quantities a, b, c, &c. enters the same number of times into the coefficient of the same term, what has here been proved of the last used is applicable to each.

8. From the last article the number of members in the several coefficients of any equation may be determined. For if we put $s =$ the number of times each quantity is found in a coefficient, $n =$ the number of quantities a, b, c, &c. and $p =$ the number of quantities in each member; then as a is found s times in this coefficient, b is found s times in this coefficient, &c. the number of quantities in this coefficient, with their repetitions, will be $s \times n$, and as p expresses the number of quantities requisite for each member, the number of members in the coefficient will be $\frac{sn}{p}$.

9. Using the same notation, we can, by the last two articles, calculate the number of members in the next coefficient. For as $\frac{sn}{p}$ expresses the number of members in the abovementioned coefficient, and s the number of times each quantity is found in it, $\frac{sn}{p} - s =$ the number of times each is not found in it. By the 6th article, therefore, a will be found $\frac{sn}{p} - s$ times, b will be found $\frac{sn}{p} - s$ times, &c. in the next coefficient, and $\overline{\frac{sn}{p} - s} \times n$ $= \frac{sn^2 - psn}{p} =$ the number of quantities, with their repetitions, in it. But as the number of quantities in each member of a coefficient is 1 less than the number in each member of the coefficient next following, each member of the coefficient

whose number of members we are now calculating will have in it $p + 1$ number of quantities. Consequently $\frac{sn^2 - psn}{p \times \overline{p+1}} = \frac{sn}{p} \times \frac{n-p}{p+1} =$ the number of members of the coefficient next after that whose number of members is $\frac{sn}{p}$, as in the last article.

The same conclusion may be obtained in the following manner. Let $m =$ the number of members in a coefficient, $p =$ the number of quantities in each member, and $n =$ the number of quantities a, b, c, &c. Then will mp express the number of quantities with their repetitions in this coefficient, and $\frac{mp}{n}$ the number of times each quantity is found in it. Hence, as each quantity is only found once in the same member, $m - \frac{mp}{n} =$ the number of times each is not found in this coefficient, and is therefore equal to the number of times each is found in the next coefficient, according to the 6th article. The number of quantities, therefore, with their repetitions, in the next coefficient is expressed by $\overline{m - \frac{mp}{n}} \times n = mn - mp$; and as the number of quantities in each of its members is denoted by $p + 1$, the number of its members is expressed by $\frac{mn - mp}{p+1} = m \times \frac{n-p}{p+1}$.

10. The binomial theorem, as far as it relates to the raising of integral powers, easily follows from the foregoing articles. For if all the quantities a, b, c, &c. used in the multiplication in the 3d article, be equal to one another, and consequently each equal to a, each of the members in any coefficient will become a power of a; and each term in an equation will consist of a power of a multiplied into a power of x, having such a numeral coefficient prefixed as expresses the num-

ber of members in the coefficient, when exhibited in the manner of the 3d article. And as n expressed the number of quantities a, b, c, &c. used in the multiplication, when each of these quantities is equal to a, it will denote the power of the binomial $x + a$.

Hence, if m denote the numeral coefficient of any term of the nth power of $x + a$, and p the exponent of a in that term, the numeral coefficient of the next term will be expressed by $m \times \frac{n-p}{p+1}$, as is evident from the last article.

11. It is manifest from the 3d article that $x + a$ being raised to the nth power, the series, without the numeral coefficients, will be $x^n + a\,x^{n-1} + a^2\,x^{n-2} + a^3\,x^{n-3} +$, &c. and as the coefficient of the first term is 1, and of the second n, from the general expression in the last article $\overline{x+a}|^n = x^n + n\,a\,x^{n-1} + n \times \frac{n-1}{2}\,a^2\,x^{n-2} + n \times \frac{n-1}{2} \times \frac{n-2}{3}\,a^3\,x^{n-3} +$, &c.

12. If equations be generated from $\overline{x-a} \times \overline{x-b} \times \overline{x-c} \times \overline{x-d}$, &c. the coefficients will be the same, excepting the signs, as those which result from $\overline{x+a} \times \overline{x+b} \times \overline{x+c} \times \overline{x+d}$, &c. in the 3d article; and as $- \times -$ gives $+$, but $- \times - \times -$ gives $-$, the coefficients, in equations generated from $\overline{x-a} \times \overline{x-b} \times \overline{x-c} \times \overline{x-d}$, &c. whose members have each an even number of quantities will have the sign $+$, but coefficients whose members have each an odd number of quantities will have the sign $-$. And hence it is evident that $\overline{x-a}|^n = x^n - n\,a\,x^{n-1} + n \times \frac{n-1}{2}\,a^2\,x^{n-2} - n \times \frac{n-1}{2} \times \frac{n-2}{3}\,a^3\,x^{n-3} +$, &c.

13. Having thus investigated the binomial theorem, as far

as it relates to the raising of integral powers, I proceed to demonstrate, by the principles of multiplication, the most general case; *viz.* that $\overline{x+z}|^{\frac{n}{r}} = x^{\frac{n}{r}} + \frac{n}{r} z x^{\frac{n}{r}-1} + \frac{n}{r} \times \frac{\frac{n}{r}-1}{2} z^2 x^{\frac{n}{r}-2}$ +, &c. This will clearly appear after it has been proved that if the series $x^{\frac{n}{r}} + \frac{n}{r} z x^{\frac{n}{r}-1} + \frac{n}{r} \times \frac{\frac{n}{r}-1}{2} z^2 x^{\frac{n}{r}-2} + \frac{n}{r} \times \frac{\frac{n}{r}-1}{2} \times \frac{\frac{n}{r}-2}{3} z^3 x^{\frac{n}{r}-3}$ +, &c. be multiplied by the series $x^{\frac{1}{r}} + \frac{1}{r} z x^{\frac{1}{r}-1} + \frac{1}{r} \times \frac{\frac{1}{r}-1}{2} z^2 x^{\frac{1}{r}-2} + \frac{1}{r} \times \frac{\frac{1}{r}-1}{2} \times \frac{\frac{1}{r}-2}{3} z^3 x^{\frac{1}{r}-3}$ +, &c. the product will be $x^{\frac{n+1}{r}} + \frac{n+1}{r} z x^{\frac{n+1}{r}-1} + \frac{n+1}{r} \times \frac{\frac{n+1}{r}-1}{2} z^2 x^{\frac{n+1}{r}-2} + \frac{n+1}{r} \times \frac{\frac{n+1}{r}-1}{2} \times \frac{\frac{n+1}{r}-2}{3} z^3 x^{\frac{n+1}{r}-3}$ +, &c. Or, which is the same thing, after it has been proved that if the series $x^{\frac{n}{r}} + \frac{n}{r} z x^{\frac{n}{r}-1} + \frac{n}{r} \times \frac{n-r}{2r} z^2 x^{\frac{n-2r}{r}} + \frac{n}{r} \times \frac{n-r}{2r} \times \frac{n-2r}{3r} z^3 x^{\frac{n-3r}{r}}$ +, &c. be multiplied by the series $x^{\frac{1}{r}} + \frac{1}{r} z x^{\frac{1-r}{r}} + \frac{1}{r} \times \frac{1-r}{2r} z^2 x^{\frac{1-2r}{r}} + \frac{1}{r} \times \frac{1-r}{2r} \times \frac{1-2r}{3r} z^3 x^{\frac{1-3r}{r}}$ +, &c. the product will be $x^{\frac{n+1}{r}} + \frac{n+1}{r} x z^{\frac{n+1-r}{r}} + \frac{n+1}{r} \times \frac{n+1-r}{2r} z^2 x^{\frac{n+1-2r}{r}} + \frac{n+1}{r} \times \frac{n+1-r}{2r} \times \frac{n+1-2r}{3r} z^3 x^{\frac{n+1-3r}{r}}$ +, &c.

14. Upon multiplying the two last series into one another, to obtain a foundation for the demonstration in view, the same powers of x and z, which arise in the multiplication, being

placed under one another, the products will stand as below; the first two lines immediately following being considered as the multiplicand and the multiplier respectively.

$$x^{\frac{n}{r}} + \frac{n}{r} z x^{\frac{n-r}{r}} + \frac{n}{r} \times \frac{n-r}{2r} z^2 x^{\frac{n-2r}{r}} + \frac{n}{r} \times \frac{n-r}{2r} \times \frac{n-2r}{3r} z^3 x^{\frac{n-3r}{r}} +, \&c.$$

$$x^{\frac{1}{r}} + \frac{1}{r} z x^{\frac{1-r}{r}} + \frac{1}{r} \times \frac{1-r}{2r} z^2 x^{\frac{1-2r}{r}} + \frac{1}{r} \times \frac{1-r}{2r} \times \frac{1-2r}{3r} z^3 x^{\frac{1-3r}{r}} +, \&c.$$

$$x^{\frac{n+1}{r}} + \frac{n}{r} z x^{\frac{n+1-r}{r}} + \frac{n}{r} \times \frac{n-r}{2r} z^2 x^{\frac{n+1-2r}{r}} + \frac{n}{r} \times \frac{n-r}{2r} \times \frac{n-2r}{3r} z^3 x^{\frac{n+1-3r}{r}} +, \&c.$$

$$\frac{1}{r} z x^{\frac{n+1-r}{r}} + \frac{1}{r} \times \frac{n}{r} z^2 x^{\frac{n+1-2r}{r}} + \frac{1}{r} \times \frac{n}{r} \times \frac{n-r}{2r} z^3 x^{\frac{n+1-3r}{r}} +, \&c.$$

$$+ \frac{1}{r} \times \frac{1-r}{2r} z^2 x^{\frac{n+1-2r}{r}} + \frac{1}{r} \times \frac{1-r}{2r} \times \frac{n}{r} z^3 x^{\frac{n+1-3r}{r}} +, \&c.$$

$$\frac{1}{r} \times \frac{1-r}{2r} \times \frac{1-2r}{3r} z^3 x^{\frac{n+1-3r}{r}} +, \&c.$$

Now, in order to establish the laws of arrangement upon clear and general principles, it is necessary to observe these particulars. 1st. The exponents of the terms, both in the multiplicand and multiplier, are in arithmetical progression; they have the same denominator r, and r is also the common difference in the numerators of each progression. 2d. The multiplicand being multiplied by $x^{\frac{1}{r}}$, the first term in the multiplier, gives the first horizontal line of products; and consequently the exponents in this line are obtained from the exponents in the multiplicand by adding 1 to the numerators. The numerators, therefore, of the exponents of this line are also in arithmetical progression; and under this the other lines of products are to be arranged, so that terms which have the same exponents may come under one another. 3d. The coefficients being neglected, if any term in the multiplicand be denoted by $z^q x^{\frac{n-qr}{r}}$, the term of the multiplier immediately under will be expressed by $z^q x^{\frac{1-qr}{r}}$, according to the nature of the two series; and upon multiplying the first term of the multiplicand by this term of the multiplier, the product will be $z^q x^{\frac{n+1-qr}{r}}$, which is equal to that term of the multiplicand immediately over that in the multiplier, after 1 is added to the numerator of the exponent of x. And the other terms in the multiplicand, successively to the right hand, being multiplied by the same term of the multiplier, the terms will be $z^{q+1} x^{\frac{n+1-qr-r}{r}}$, $z^{q+2} x^{\frac{n+1-qr-2r}{r}}$, $z^{q+3} x^{\frac{n+1-qr-3r}{r}}$, &c. in arithmetical progression, which are equal to those terms of the multiplicand immediately over them, after the numerators of the exponents of x are increased by 1. And from hence a general rule is obtained for the arrangement of any horizontal line of products. For when the first term in the multiplicand is multiplied by a term in the multiplier, the product is placed immediately under that term of the multiplier; and the products which arise from multiplying the other terms of the multiplicand, successively towards the right, by the same term of the multiplier, are placed successively towards the right of the first mentioned product.

15. The several products, therefore, arranged under one another in a perpendicular line, arise in the following manner. The first arises from multiplying the term in the multiplicand directly over it into the first term in the multiplier. Thus $\frac{n}{r} \times \frac{n-r}{2r} \times \frac{n-2r}{3r} z^3 x^{\frac{n+1-3r}{r}}$ is the product of $\frac{n}{r} \times$

$\frac{n-r}{2r}\times\frac{n-2r}{3r}z^3x^{\overline{\frac{n-3r}{r}}}$, the term of the multiplicand directly over it, into $x^{\frac{1}{r}}$, the first term in the multiplier. The second term in the perpendicular line of products is obtained by multiplying that term of the multiplicand in the next perpendicular line towards the left, by the second term of the multiplier. Thus $\frac{1}{r}\times\frac{n}{r}\times\frac{n-r}{2r}z^3x^{\overline{\frac{n+1-3r}{r}}}$, is the product of $\frac{n}{r}\times\frac{n-r}{2r}z^2x^{\overline{\frac{n-2r}{r}}}$ into $\frac{1}{r}zx^{\overline{\frac{1-r}{r}}}$. And in general, if p be put for a number denoting the place of a term in the perpendicular line of products, and if the terms in the multiplicand be supposed to be numbered, beginning with that directly above the perpendicular line of products under consideration, and reckoning towards the left hand; and if the terms in the line of the multiplier be numbered, beginning with $x^{\frac{1}{r}}$, and reckoning towards the right, then the product whose place is p will arise from the multiplication of that term in the multiplicand whose place is denoted by p into that term in the multiplier whose place is also denoted by p. The observations in this and the last article are evidently general; being applicable to any extent to which the series in the multiplicand and multiplier may be carried.

16. The laws of arrangement being thus established by the exponents, the summation of the coefficients, in any perpendicular line of products, is next to be attended to. And in order to do this, with as little embarrassment as possible, put $A=n$, $B=n\times\overline{n-r}$, $C=n\times\overline{n-r}\times\overline{n-2r}$, $D=n\times\overline{n-1}\times\overline{n-2r}\times\overline{n-3r}$, &c. and put $a=1$, $b=1\times\overline{1-r}$, $c=1$

$\times \overline{1-r} \times \overline{1-2r}$, $d = 1 \times \overline{1-r} \times \overline{1-2r} \times \overline{1-3r}$, &c. Moreover, put $\alpha = 1$, $\beta = 1 \times 2$, $\gamma = 1 \times 2 \times 3$, $\delta = 1 \times 2 \times 3 \times 4$, &c. and then the multiplicand, multiplier, and products will stand in the following manner, the powers of x and z being omitted.

$$1 + \frac{A}{\alpha r} + \frac{B}{\beta r^2} + \frac{C}{\gamma r^3} + \frac{D}{\delta r^4} + \frac{E}{\varepsilon r^5} + \frac{F}{\zeta r^6} + \frac{G}{\eta r^7} +, \&c.$$

$$1 + \frac{a}{\alpha r} + \frac{b}{\beta r^2} + \frac{c}{\gamma r^3} + \frac{d}{\delta r^4} + \frac{e}{\varepsilon r^5} + \frac{f}{\zeta r^6} + \frac{g}{\eta r^7} +, \&c.$$

$$1 + \frac{A}{\alpha r} + \frac{B}{\beta r^2} + \frac{C}{\gamma r^3} + \frac{D}{\delta r^4} + \frac{E}{\varepsilon r^5} + \frac{F}{\zeta r^6} + \frac{G}{\eta r^7} +, \&c.$$

$$\frac{a}{\alpha r} + \frac{aA}{\alpha\alpha r^2} + \frac{aB}{\alpha\beta r^3} + \frac{aC}{\alpha\gamma r^4} + \frac{aD}{\alpha\delta r^5} + \frac{aE}{\alpha\varepsilon r^6} + \frac{aF}{\alpha\zeta r^7} +, \&c.$$

$$\frac{b}{\beta r^2} + \frac{bA}{\beta\alpha r^3} + \frac{bB}{\beta\beta r^4} + \frac{bC}{\beta\gamma r^5} + \frac{bD}{\beta\delta r^6} + \frac{bE}{\beta\varepsilon r^7} +, \&c.$$

$$\frac{c}{\gamma r^3} + \frac{cA}{\gamma\alpha r^4} + \frac{cB}{\gamma\beta r^5} + \frac{cC}{\gamma\gamma r^6} + \frac{cD}{\gamma\delta r^7} +, \&c.$$

$$\frac{d}{\delta r^4} + \frac{dA}{\delta\alpha r^5} + \frac{dB}{\delta\beta r^6} + \frac{dC}{\delta\gamma r^7} +, \&c.$$

$$\frac{e}{\varepsilon r^5} + \frac{eA}{\varepsilon\alpha r^6} + \frac{eB}{\varepsilon\beta r^7} +, \&c.$$

$$\frac{f}{\zeta r^6} + \frac{fA}{\zeta\alpha r^7} +, \&c.$$

$$\frac{g}{\eta r^7} +, \&c.$$

Now the object in view, with respect to the coefficients, is to prove that the perpendicular lines of products will be, beginning at 1 and reckoning towards the right hand, equal to $1, \frac{n+1}{r}, \frac{n+1}{r} \times \frac{n+1-r}{2r}, \frac{n+1}{r} \times \frac{n+1-r}{2r} \times \frac{n+1-2r}{3r}$, &c. respectively: and this will be fully demonstrated when we have proved that all the terms of products in any perpendicular line, in which the exponent of r in the denominators is t, being multiplied by $\frac{n+1-tr}{\overline{t+1} \times r}$ are equal to the whole of the next perpendicular line of products towards the right hand.

To do this in a manner applicable to any part of the series concerned, and to avoid numeral coefficients, which would obscure and encumber the general reasoning, it is necessary to find the value of the numerator of $\frac{n+1-tr}{\overline{t+1} \times r}$ in terms of A, B, C, D, &c. and of *a*, *b*, *c*, *d*, &c. and to ascertain the relative values of α, β, γ, δ, &c. and that we may do this with due precision and perspicuity, it is proper to fix upon two contiguous perpendicular lines of products.

17. Let the lines be those which have in their first terms F and G respectively, and then $n+1-tr = \frac{G}{F}+1 = \frac{F}{E}+1 - r = \frac{E}{D}+1-2r = \frac{D}{C}+1-3r = \frac{C}{B}+1-4r = \frac{B}{A}+1-5r = \frac{A}{1}+1-6r$; and therefore, according to the substitution in the last article, $n+1-tr = \frac{G}{F}+a = \frac{F}{E}+\frac{b}{a} = \frac{E}{D}+\frac{c}{b} = \frac{D}{C}+\frac{d}{c} = \frac{C}{B}+\frac{e}{d} = \frac{B}{A}+\frac{f}{e} = \frac{A}{1}+\frac{g}{f}$. Now the first of the two contiguous perpendicular lines fixed upon being multiplied by these values, *viz.* the first term being multiplied by the first value, the second term by the second value, &c. and the denominators ζr^6, $\alpha \varepsilon r^6$, &c. in the first line, and ηr^7 $\alpha \zeta r^7$, &c. in the other being omitted, the two lines will be as represented in the following columns.

The first of the two contiguous lines multiplied as mentioned above.	The last of the two contiguous lines as mentioned above.
	G
$F \times \overline{\frac{G}{F} + a} = G + aF$	aF
$aE \times \overline{\frac{F}{E} + \frac{b}{a}} = aF + bE$	bE
$bD \times \overline{\frac{E}{D} + \frac{c}{b}} = bE + cD$	cD
$cC \times \overline{\frac{D}{C} + \frac{d}{c}} = cD + dC$	dC
$dB \times \overline{\frac{C}{B} + \frac{e}{d}} = dC + eB$	eB
$eA \times \overline{\frac{B}{A} + \frac{f}{e}} = eB + fA$	fA
$f \times \overline{\frac{A}{1} + \frac{g}{f}} = fA + g$	g

The proper denominators being annexed to these terms, and v being put for $t + 1$, it now remains to be proved that

$$\frac{G+aF}{\zeta r^6 \times vr} + \frac{aF+bE}{\alpha\varepsilon r^6 \times vr} + \frac{bE+cD}{\beta\delta r^6 \times vr} + \frac{cD+dC}{\gamma\gamma r^6 \times vr} + \frac{dC+eB}{\delta\beta r^6 \times vr} + \frac{eB+fA}{\varepsilon\alpha r^6 \times vr} + \frac{fA+g}{\zeta r^6 \times vr} = \frac{G}{\eta r^7} + \frac{aF}{\alpha\zeta r^7} + \frac{bE}{\beta\varepsilon r^7} + \frac{cD}{\gamma\delta r^7} + \frac{dC}{\delta\gamma r^7} + \frac{eB}{\varepsilon\beta r^7} + \frac{fA}{\zeta\alpha r^7} + \frac{g}{\eta r^7}.$$

18. The relative values, therefore, of α, β, γ, &c. next claim our attention; and from the nature of the series, $\frac{\eta}{v} = \zeta$, $\frac{\zeta}{v-1} = \varepsilon$, $\frac{\varepsilon}{v-2} = \delta$, $\frac{\delta}{v-3} = \gamma$, $\frac{\gamma}{v-4} = \beta$, $\frac{\beta}{v-5} = \alpha$, $\frac{\alpha}{v-6} = 1$. Also $1 = \alpha$, $\frac{\beta}{2} = \alpha$, $\frac{\gamma}{3} = \beta$, $\frac{\delta}{4} = \gamma$, $\frac{\varepsilon}{5} = \delta$, $\frac{\zeta}{6} = \varepsilon$, $\frac{\eta}{7} = \zeta$. As the powers of r in the equation, asserted in the end of the last article, are the same in all the terms, they may be neglected; the only thing necessary is to reduce ζ, $\alpha\varepsilon$, $\beta\delta$, $\gamma\gamma$, $\delta\beta$, $\varepsilon\alpha$, ζ, the denominators of the first side, to η, $\alpha\zeta$, $\beta\varepsilon$, $\gamma\delta$, $\delta\gamma$, $\varepsilon\beta$, $\zeta\alpha$, η, the de-

nominators of the second, and in such a manner as to make the parts on the first side, which have the same numerators, unite: thus the part $\frac{G}{\zeta}$ must be reduced to the denominator η; the parts $\frac{aF}{\zeta v} + \frac{aF}{\alpha\varepsilon v}$ must be reduced to the same denominator $\alpha\zeta$; the parts $\frac{bE}{\alpha\varepsilon v} + \frac{bE}{\beta\delta v}$ to the same denominators $\beta\varepsilon$, &c.

First line.	Second line.
$\frac{G + aF}{\zeta v}$	$\frac{G}{\eta}$
$\frac{aF + bE}{\alpha\varepsilon v}$	$\frac{aF}{\alpha\zeta}$
$\frac{bE + cD}{\beta\delta v}$	$\frac{bE}{\beta\varepsilon}$
$\frac{cD + dC}{\gamma\gamma v}$	$\frac{cD}{\gamma\delta}$
$\frac{dC + eB}{\delta\beta v}$	$\frac{dC}{\delta\gamma}$
$\frac{eB + fA}{\varepsilon\alpha v}$	$\frac{eB}{\varepsilon\beta}$
$\frac{fA + g}{\zeta v}$	$\frac{fA}{\zeta\alpha}$
	$\frac{g}{\eta}$

Now, upon examining the two lines as represented in the columns in the margin, a general rule for this reduction presents itself. For the denominators, exclusive of v in the first column, proceed in the following regular manner, which is not peculiar to the perpendicular lines now under examination, but is the same in any two contiguous lines in any period of the multiplication exhibited in the 16th article. The first and last denominator, in each column, consists of a single letter, as ζ in the first, and η in the second, of these we have selected for illustration. The second denominator consists of the next lower letter to the highest multiplied into α, as $\alpha\varepsilon$ in the first column, and $\alpha\zeta$ in the second. The third denominator consists of the second lower letter to the highest multiplied into β, as $\beta\delta$ in the first column, and $\beta\varepsilon$ in the second; and the same gradation is observed in the other denominators. Now each term in the first column has two members in the numerator, and to make these unite with the terms in the second column, the first member must have the same denominator with that term in the second column in the same horizontal line; and the second member must have the same denominator with that term in the second column on the next lower horizontal line. For the first member, therefore, the second letter in the denominator must be raised to the next higher letter by substitution; and for the second member the first letter in the denominator must be raised to the next higher by substitution. For each denominator, therefore, in the first column two equal values must be found accordingly, and the first value must be put under the first member, and the second value under the second member of the numerators. Hence the values for the denominators of the first line will be obtained in the following manner, from the equations in the beginning of this article. For the first term $\zeta v = \frac{\eta}{v} \times v = \eta = \alpha \zeta v$, for $\alpha = 1$; for the second term $\alpha \varepsilon v = \alpha \times \frac{\zeta}{v-1} \times v = \frac{\alpha \zeta v}{v-1} = \frac{\beta}{2} \times \varepsilon v = \frac{\beta \varepsilon v}{2}$; for the third, $\beta \delta v = \beta \times \frac{\varepsilon}{v-2} \times v = \frac{\beta \varepsilon v}{v-2} = \frac{\gamma}{3} \times \delta v = \frac{\gamma \delta v}{3}$; for the fourth, $\gamma \gamma v = \gamma \times \frac{\delta}{v-3} \times v = \frac{\gamma \delta v}{v-3} = \frac{\delta}{4} \times \gamma v = \frac{\delta \gamma v}{4}$; for the fifth, $\delta \beta v = \delta \times \frac{\gamma}{v-4} \times v = \frac{\delta \gamma v}{v-4} = \frac{\varepsilon}{5} \times \beta v = \frac{\varepsilon \beta v}{5}$; for the sixth, $\varepsilon \alpha v = \varepsilon \times \frac{\beta}{v-5} \times v = \frac{\varepsilon \beta v}{v-5} = \frac{\zeta}{6} \times \alpha v = \frac{\zeta \alpha v}{6}$; for the last, $\zeta v = \zeta \times \frac{\alpha}{v-6} \times v = \frac{\zeta \alpha v}{v-6} = \frac{\eta}{7} \times v = \eta$. It therefore follows that

$$\frac{G+aF}{\zeta v} = \frac{G}{\eta} + \frac{aF}{\alpha\zeta v}$$

$$\frac{aF+bE}{\alpha\varepsilon v} = \ldots \frac{aF\times\overline{v-1}}{\alpha\zeta v} + \frac{bE+2}{\beta\varepsilon v}$$

$$\frac{bE+cD}{\beta\delta v} = \ldots\ldots \frac{bE\times\overline{v-2}}{\beta\varepsilon v} + \frac{cD\times 3}{\gamma\delta v}$$

$$\frac{cD+dC}{\gamma\gamma v} = \ldots\ldots\ldots \frac{cD\times\overline{v-3}}{\gamma\delta v} + \frac{dC\times 4}{\delta\gamma v}$$

$$\frac{dC+eB}{\delta\beta v} = \ldots\ldots\ldots\ldots \frac{dC\times\overline{v-4}}{\delta\gamma v} + \frac{eB\times 5}{\varepsilon\beta v}$$

$$\frac{eB+fA}{\varepsilon\alpha v} = \ldots\ldots\ldots\ldots\ldots \frac{eB\times\overline{v-5}}{\varepsilon\beta v} + \frac{fA\times 6}{\zeta\alpha v}$$

$$\frac{fA+g}{\zeta v} = \ldots\ldots\ldots\ldots\ldots\ldots \frac{fA\times\overline{v-6}}{\zeta\alpha v} + \frac{g}{\eta}$$

And consequently $\frac{G+aF}{\zeta v} + \frac{aF+eB}{\alpha\varepsilon v} + \frac{bE+cD}{\beta\delta v} + \frac{cD+dC}{\gamma\gamma v} + \frac{dC+eB}{\delta\beta v} + \frac{eB+fA}{\varepsilon\alpha v} + \frac{fA+g}{\zeta v} = \frac{G}{\eta} + \frac{aF}{\alpha\zeta} + \frac{bE}{\beta\varepsilon} + \frac{cD}{\gamma\delta} + \frac{dC}{\delta\gamma} + \frac{eB}{\varepsilon\beta} + \frac{fA}{\zeta\alpha} + \frac{g}{\eta}$.

19. This being proved from the relations between the two contiguous perpendicular lines, and these relations being the same between any two perpendicular lines whatever (for they are as regular and certain as the laws of continuation in the multiplicand and multiplier with which we set out in the 13th article) it follows that if $\frac{pz^m x^{\frac{n+1-mr}{r}}}{qr^m}$ express the whole of any perpendicular line, the next perpendicular line to the right will be $\frac{p\times\overline{n+1-mr}\,z^{m+1}x^{\frac{n+1-\overline{m+1}r}{r}}}{q\times\overline{m+1}\,r^{m+1}}$. And therefore the series $x^{\frac{n}{r}} + \frac{n}{r}zx^{\frac{n}{r}-1} + \frac{n}{r}\times\frac{\frac{n}{r}-1}{2}z^2x^{\frac{n}{r}-2} + \frac{n}{r}\times\frac{\frac{n}{r}-1}{2}\times$

$\frac{\frac{n}{r}-2}{3} z^3 x^{\frac{n}{r}-3}$ +, &c. being multiplied by the series $x^{\frac{1}{r}} + \frac{1}{r} z x^{\frac{1}{r}-1} + \frac{1}{r} \times \frac{\frac{1}{r}-1}{2} z^2 x^{\frac{1}{r}-2} + \frac{1}{r} \times \frac{\frac{1}{r}-2}{2} \times \frac{\frac{1}{r}-1}{3} z^3 x^{\frac{1}{r}-3}$ +, &c. the product will be $x^{\frac{n+1}{r}} + \frac{n+1}{r} z x^{\frac{n+1}{r}-1} + \frac{n+1}{r} \times \frac{\frac{n+1}{r}-1}{2} z^2 x^{\frac{n+1}{r}-2} + \frac{n+1}{r} \times \frac{\frac{n+1}{r}-1}{2} \times \frac{\frac{n+1}{r}-2}{3} z^3 x^{\frac{n+1}{r}-3}$ +, &c.

20. From hence it follows, that $\overline{x+z}|^{\frac{n}{r}} = x^{\frac{n}{r}} + \frac{n}{r} z x^{\frac{n}{r}-1} + \frac{n}{r} \times \frac{\frac{n}{r}-1}{2} z^2 x^{\frac{n}{r}-2} + \frac{n}{r} \times \frac{\frac{n}{r}-1}{2} \times \frac{\frac{n}{r}-2}{3} z^3 x^{\frac{n}{r}-3}$ +, &c. For as n in the last article may denote any number whatever, the square of the series $x^{\frac{1}{r}} + \frac{1}{r} z x^{\frac{1}{r}-1} + \frac{1}{r} \times \frac{\frac{1}{r}-1}{2} z^2 x^{\frac{1}{r}-2} + \frac{1}{r} \times \frac{\frac{1}{r}-1}{2} \times \frac{\frac{1}{r}-2}{3} z^3 x^{\frac{1}{r}-3}$ +, &c. will be $x^{\frac{2}{r}} + \frac{2}{r} z x^{\frac{2}{r}-1} + \frac{2}{r} \times \frac{\frac{2}{r}-1}{2} z^2 x^{\frac{2}{r}-2} + \frac{2}{r} \times \frac{\frac{2}{r}-1}{2} \times \frac{\frac{2}{r}-2}{3} z^3 x^{\frac{2}{r}-3}$ +, &c. and this being multiplied by $x^{\frac{1}{r}} + \frac{1}{r} z x^{\frac{1}{r}-1} + \frac{1}{r} \times \frac{\frac{1}{r}-1}{2} z^2 x^{\frac{1}{r}-2} + \frac{1}{r} \times \frac{\frac{1}{r}-1}{2} \times \frac{\frac{1}{r}-2}{3} z^3 x^{\frac{1}{r}-3}$ +, &c. the product will be $x^{\frac{3}{r}} + \frac{3}{r} z x^{\frac{3}{r}-1} + \frac{3}{r} \times \frac{\frac{3}{r}-1}{2} z^2 x^{\frac{3}{r}-2} + \frac{3}{r} \times \frac{\frac{3}{r}-1}{2} \times \frac{\frac{3}{r}-2}{3} z^3 x^{\frac{3}{r}-3}$ +, &c. 1 being added to the numerator of the fraction of which r is the denominator, upon every multiplication. The

*n*th power, therefore, of the series $x^{\frac{1}{r}} + \frac{1}{r} z x^{\frac{1}{r}-1} + \frac{1}{r} \times \frac{\frac{1}{r}-1}{2} z^2 x^{\frac{1}{r}-2} + \frac{1}{r} \times \frac{\frac{1}{r}-1}{2} \times \frac{\frac{1}{r}-2}{3} z^3 x^{\frac{1}{r}-3} +$, &c. is equal to $x^{\frac{n}{r}} + \frac{n}{r} z x^{\frac{n}{r}-1} + \frac{n}{r} \times \frac{\frac{n}{r}-1}{2} z^2 x^{\frac{n}{r}-2} + \frac{n}{r} \times \frac{\frac{n}{r}-1}{2} \times \frac{\frac{n}{r}-2}{3} z^3 x^{\frac{n}{r}-3} +$, &c. and this series, when *n* is equal to *r*, becomes equal to $x + z$. Hence $\overline{x+z}|^{\frac{1}{r}} = x^{\frac{1}{r}} + \frac{1}{r} z x^{\frac{1}{r}-1} + \frac{1}{r} \times \frac{\frac{1}{r}-1}{2} z^2 x^{\frac{1}{r}-2} + \frac{1}{r} \times \frac{\frac{1}{r}-1}{2} \times \frac{\frac{1}{r}-2}{3} z^3 x^{\frac{1}{r}-3} +$, &c. and consequently $\overline{x+z}|^{\frac{n}{r}} = x^{\frac{n}{r}} + \frac{n}{r} z x^{\frac{n}{r}-1} + \frac{n}{r} \times \frac{\frac{n}{r}-1}{2} z^2 x^{\frac{n}{r}-2} + \frac{n}{r} \times \frac{\frac{n}{r}-1}{2} \times \frac{\frac{n}{r}-2}{3} z^3 x^{\frac{n}{r}-3} +$, &c.

21. From the preceding method of investigating the theorem it also follows that $\overline{x-z}|^{\frac{n}{r}} = x^{\frac{n}{r}} - \frac{n}{r} z x^{\frac{n}{r}-1} + \frac{n}{r} \times \frac{\frac{n}{r}-1}{2} z^2 x^{\frac{n}{r}-2} - \frac{n}{r} \times \frac{\frac{n}{r}-1}{2} \times \frac{\frac{n}{r}-2}{3} z^3 x^{\frac{n}{r}-3} +$, &c. For the series $x^{\frac{n}{r}} - \frac{n}{r} z x^{\frac{n}{r}-1} + \frac{n}{r} \times \frac{\frac{n}{r}-1}{2} z^2 x^{\frac{n}{r}-2} - \frac{n}{r} \times \frac{\frac{n}{r}-1}{2} \times \frac{\frac{n}{r}-2}{3} z^3 x^{\frac{n}{r}-3} +$, &c. being multiplied by $x^{\frac{1}{r}} - \frac{1}{r} z x^{\frac{1}{r}-1} + \frac{1}{r} \times \frac{\frac{1}{r}-1}{2} z^2 x^{\frac{1}{r}-2} - \frac{1}{r} \times \frac{\frac{1}{r}-1}{2} \times \frac{\frac{1}{r}-2}{3} z^3 x^{\frac{1}{r}-3} +$, &c. the product will stand as below, the laws of arrangement being the same as those established in the 14th and 15th articles.

$$x^{\frac{n}{r}} - \frac{n}{r} z x^{\frac{n-r}{r}} + \frac{n}{r} \times \frac{n-r}{2r} z^2 x^{\frac{n-2r}{r}} - \frac{n}{r} \times \frac{n-r}{2r} \times \frac{n-2r}{3r} z^3 x^{\frac{n-3r}{r}} +, \&c.$$

$$x^{\frac{1}{r}} - \frac{1}{r} z x^{\frac{1-r}{r}} + \frac{1}{r} \times \frac{1-r}{2r} z^2 x^{\frac{1-2r}{r}} - \frac{1}{r} \times \frac{1-r}{2r} \times \frac{1-2r}{3r} z^3 x^{\frac{1-3r}{r}} +, \&c.$$

$$x^{\frac{n+1}{r}} - \frac{1}{r} z x^{\frac{n+1-r}{r}} + \frac{n}{r} \times \frac{n-r}{2r} z^2 x^{\frac{n+1-2r}{r}} - \frac{n}{r} \times \frac{n-r}{2r} \times \frac{n-2r}{3r} z^3 x^{\frac{n+1-3r}{r}} +, \&c.$$

$$- \frac{1}{r} z x^{\frac{n+1-r}{r}} + \frac{1}{r} \times \frac{n}{r} z^2 x^{\frac{n+1-2r}{r}} - \frac{1}{r} \times \frac{n}{r} \times \frac{n-r}{2r} z^3 x^{\frac{n+1-3r}{r}} +, \&c.$$

$$\frac{1}{r} \times \frac{1-r}{2r} z^2 x^{\frac{n+1-2r}{r}} - \frac{1}{r} \times \frac{1-r}{2r} \times \frac{n}{r} z^3 x^{\frac{n+1-3r}{r}} +, \&c.$$

$$- \frac{1}{r} \times \frac{1-r}{2r} \times \frac{1-2r}{3r} z^3 x^{\frac{n+1-3r}{r}} +, \&c.$$

And from hence it is evident that these perpendicular lines differ from those in the 14th article, in the signs only; the signs in the above being alternately + and —. It therefore may be demonstrated, as in the foregoing articles, that $\overline{x-z}|^{\frac{n}{r}}$

$$= x^{\frac{n}{r}} - \frac{n}{r} z x^{\frac{n}{r}-1} + \frac{n}{r} \times \frac{\frac{n}{r}-1}{2} z^2 x^{\frac{n}{r}-2} - \frac{n}{r} \times \frac{\frac{n}{r}-1}{2} \times \frac{\frac{n}{r}-2}{3} z^3 x^{\frac{n}{r}-3} +, \&c.$$

INDEX.

From the Press of
W. BULMER & Co.
Cleveland-Row, St. James's.

ERRATA.

PART I.

Page 147 line 13, *for* 25, *read* 2,5.

PART II.

Page 298 line 2, insert *the Rev.* after *By*.
Page 302 line 2, *insert* in *after* units.
Page 318 line 2 from bottom, for $p \times \overline{n+1-mrz}^{m+1}$, read $p \times \overline{n+1-mrz}^{m+1}$.
Page 343 line 5 from bottom, for *f, g*, read *e, f*.
line 3 from bottom, for 3. *c*, read 3. *b*.
line 2 from bottom, for *f, g*, read *e, f*.
Page 344 line 3, for *i, k*, read *h, i*.
line 4, for *g*, read *f*.
line last but one, for *g*, read *f*.
Page 345 line 4, for *i*, read *h*.
line 5, for *g*, read *h*.
Page 364 line 6, *for* rail, *read* rails.
Page 372 line 17, *dele* comma *after* shape.
Page 433 line 11, *for* in which the chains were laid off, *read* to which the chains were reduced.
Page 492 line 17, *for* in the degrees, *read* in degrees.
Page 517 line 2, *for* Direction, *read* Directions.
Page 522 line 11, *for* E and L, *read* R and W.
Page 526 line 4, *for* a degree, *read* the degree.
Page 537 line 19, *for* hopothesis, *read* hypothesis.
Page 557 line 8, *et alibi, for* Gov. Hornsby, *read* Gov. Hornby.